Hierarchical subspace models for contingency tables

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Notations on contingency tables

- Notations for m-way contingency tables
  - $I := I_1 \times \cdots \times I_m$: number of cells of an m-way table
    - I_k : number of levels for k-th variable
  - $\mathcal{I} := [I_1] \times \cdots \times [I_m]$: set of cells
    - $[I_k] = \{1, \ldots, I_k\}$
  - $\boldsymbol{i} = (i_1 i_2 \cdots i_m) \in \mathcal{I}$: each cell
  - For $D \in [m]$,
    - $\boldsymbol{i}_D$: marginal cell
    - $\mathcal{I}_D$: the set of marginal cells for D
    - I_D : the number of marginal cells for D
  - $x(\cdot)$: frequencies
  - $p(\cdot)$: cell probabilities

Space of m-way tables

- $V = \mathbb{R}^I = \mathbb{R}^{I_1 \times \cdots \times I_m}$
  - the set of m-way tables with real entries
  - I-dimensional real vector space of functions $\psi : \mathcal{I} \mapsto \mathbb{R}$

- L_D for $D \subset [m]$:
  the set of functions depending only on D marginal cells

  $$L_D = \{\psi \in V \mid \psi(i_1, \ldots, i_m) = \psi(i'_1, \ldots, i'_m) \text{ if } i_v = i'_v, \forall v \in D\}$$

  - L_D is considered as $\mathbb{R}^{I_D}$, where $I_D = \prod_{v \in D} I_v$
  - If $D = [m]$, $L_D = L_{[m]} = V$

Log affine model

- L : a linear subspace of V s.t. $1 \in L$
- Log-affine model :

$$\log p(\cdot) := \{\log p(\boldsymbol{i}), \boldsymbol{i} \in \mathcal{I}\} \in L$$

- V and L_D specify saturated models
- If $L_D \subset L$, we say that D is saturated in L

Hierarchical model

- Δ : a simplicial complex
- $\mathrm{red}\Delta$: the set of maximal elements (facets) of Δ
- Hierarchical model L_Δ :

$$\log p(\cdot) \in L_\Delta := \sum_{D \in \mathrm{red}\Delta} L_D,$$

- If $\mathrm{red}\Delta = [m]$, L_Δ is m-way saturated model

Example 1. modeling for $I \times J$ tables

- Two-way saturated model : $\mathrm{red}\Delta := \{\{1,2\}\}$

$$V = L_{\{1,2\}} \;:\; \log p_{i_1 i_2} = \alpha_{i_1} + \beta_{i_2} + \gamma_{i_1 i_2}$$

- A log-affine model L :

$$L \;:\; \log p_{i_1 i_2} = \alpha_{i_1} + \beta_{i_2} + \gamma \phi_{i_1 i_2} \;\subset V$$

  - $\phi_{i_1 i_2}$: known functions

ex 1. uniform association model : $\phi_{i_1 i_2} = i_1 i_2$

ex 2. two-way change point model (Hirotsu(1997)) :

$$\phi_{i_1 i_2} = \begin{cases} 1, & \text{if } i_1 \le I_1' < I_1 \text{ and } i_2 \le I_2' < I_2, \\ 0, & \text{otherwise,} \end{cases}$$

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Example 2. 3-way Split model

- Split model (Højsgaard (2003)):
  The conditional independence structures are different for specific values of the conditioning variables

  ex. $L = L^{i_2=1}_{\{1\}} + L^{i_2=1}_{\{3\}} + L^{i_2=2}_{\{1,3\}}$

  - $i_2 = 1$ slice : $L^{i_2=1} = L_{\{1\}} + L_{\{3\}}$
  - $i_2 = 2$ slice : $L^{i_2=2} = L_{\{1,3\}}$
  - $L \subset L_{\{1,2,3\}}$

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Main purpose

- We propose "hierarchical subspace model (HSM)" as a generalization of the hierarchical model
- The notion of HSM gives a modeling strategy of multiway tables and unifies various models of interaction effects
- In this talk we discuss HSM from a viewpoint of localization of the computation of MLE and Markov bases
- We also illustrate practical advantage of our modeling strategy with some data sets

Hierarchical subspace model

Conformality of Log-affine model

Definition (conformality)

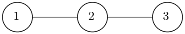
$W_1, \dots, W_K$: linear subspaces of V
$W := W_1 + \cdots + W_K$
L is *conformal* to $\{W_j\}_{j=1}^K$ if

$$L = L \cap W = (L \cap W_1) + \cdots + (L \cap W_K)$$

- $L \supset (L \cap W_1) + \cdots + (L \cap W_K)$ always holds
- The inclusion is strict in general

Ex. 3-way conditional independence model

- $W_1 := L_{\{1,2\}}$, $W_2 := L_{\{2,3\}}$
- $W = W_1 + W_2 = L_{\{1,2\}} + L_{\{2,3\}}$
- W : $\log p_{i_1 i_2 i_3} = \alpha_{i_1} + \beta_{i_2} + \gamma_{i_3} + \delta_{i_1 i_2} + \delta'_{i_2 i_3}$



- L : $\log p_{i_1 i_2 i_3} = \alpha_{i_1} + \beta_{i_2} + \gamma_{i_3} + \delta\phi_{i_1 i_2} + \delta'\psi_{i_2 i_3}$
  - $\phi_{i_1 i_2}$, $\psi_{i_2 i_3}$: known functions
  - δ, δ': free parameters
  - $L \cap W_1 = \{\alpha_{i_1} + \beta_{i_2} + \delta\phi_{i_1 i_2}\}$
  - $L \cap W_2 = \{\beta_{i_2} + \gamma_{i_3} + \delta'\psi_{i_2 i_3}\}$

$\rightarrow$ $\boxed{L = (L \cap W_1) + (L \cap W_2)}$ conformal!

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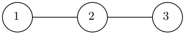

- $L': \ \log p_{i_1 i_2 i_3} = \alpha_{i_1} + \beta_{i_2} + \gamma_{i_3} + \delta(\phi_{i_1 i_2} + \psi_{i_2 i_3})$
  - $L' \cap W_1 = \{\alpha_{i_1} + \beta_{i_2}\}$
  - $L' \cap W_2 = \{\beta_{i_2} + \gamma_{i_3}\}$

$$\rightarrow \quad (L' \cap W_1) + (L' \cap W_2) = \alpha_{i_1} + \beta_{i_2} + \gamma_{i_3}$$

$$\rightarrow \quad \boxed{L' \supset (L' \cap W_1) + (L' \cap W_2)}$$

not conformal

  - Intuitively conformality represents decomposability of L

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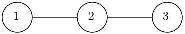

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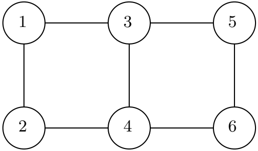
$$\rightarrow \quad \boxed{L' \supset (L' \cap W_1) + (L' \cap W_2)}$$ not conformal

  - Intuitively conformality represents decomposability of L

Terminologies on hypergraphs

- $\mathrm{red}\Delta$ is considered as a hypergraph
- A divider S of $\mathrm{red}\Delta$:
  $\exists u, v$ s.t. S is a minimal clique separator separating u and v
  - $\mathcal{S}$: the set of dividers of $\mathrm{red}\Delta$
- u and v are tightly connected:
  there is no divider separating u and v
- compact component C:
  any two vertices in C are tightly connected
  - $\mathcal{C}$: the set of maximal compact components of $\mathrm{red}\Delta$

ex. A divider : $\{3, 4\}$,
Compact components : $\{1, 2, 3, 4\}$, $\{3, 4, 5, 6\}$



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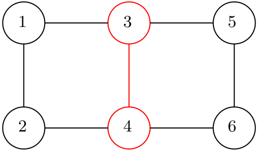

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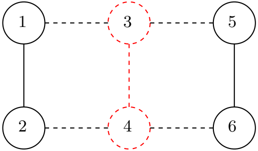

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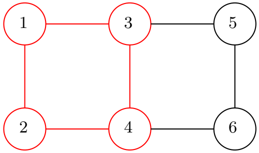

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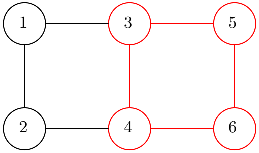

Hierarchical subspace model (HSM)

Definition (hierarchical subspace model)

L is a hierarchical subspace model of a hierarchical model L_Δ if

1. $L_S \subset L$ for each $S \in \mathcal{S}$
2. L is conformal to $\{L_C, C \in \mathcal{C}\}$

- $L_S \subset L \Rightarrow \hat{p}(\boldsymbol{i}_S) = x(\boldsymbol{i}_S)/n$
  - $\hat{p}(\boldsymbol{i}_S)$: MLE for $L \cap L_S$, $x(\boldsymbol{i}_S)$: a marginal frequency for $\boldsymbol{i}_S$
- Conformality : the same decomposability as L_Δ
  - $L_\Delta = L_{\{1,2\}} + L_{\{2,3\}}$
  - L: $\log p_{ijk} = \alpha_i + \beta_j + \gamma_k + \delta\phi_{ij} + \delta'\psi_{jk}$
    $\Rightarrow$ HSM of L_Δ
  - L': $\log p_{ijk} = \alpha_i + \beta_j + \gamma_k + \delta(\phi_{ij} + \psi_{jk})$
    $\Rightarrow$ L' is not conformal to $\{L_{\{1,2\}}, L_{\{2,3\}}\}$
    $\Rightarrow$ L' is not an HSM of L_Δ
  - Every log-affine model is an HSM of the saturated model

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  - $L_\triangle = L_{\{1,2\}} + L_{\{2,3\}}$
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MLE of HSM

- L : HSM of L_Δ
  - L has the same conditional independence structure as L_Δ
- $p(\boldsymbol{i}_C)$: cell prob for $L \cap L_C$

$$p(\boldsymbol{i}) = \frac{\prod_{C \in \mathcal{C}} p(\boldsymbol{i}_C)}{\prod_{S \in \mathcal{S}} p(\boldsymbol{i}_S)}$$

- $\hat{p}(\boldsymbol{i})$: MLE of $p(\boldsymbol{i})$
- $\hat{p}(\boldsymbol{i}_C)$: MLE for $L \cap L_C$

$$\hat{p}(\boldsymbol{i}) = \frac{\prod_{C \in \mathcal{C}} \hat{p}(\boldsymbol{i}_C)}{\prod_{S \in \mathcal{S}} \hat{p}(\boldsymbol{i}_S)} = \frac{\prod_{C \in \mathcal{C}} \hat{p}(\boldsymbol{i}_C)}{\prod_{S \in \mathcal{S}} x(\boldsymbol{i}_S)/n}$$

Decomposability of log-affine model

- So far we have discussed the definition of HSM
- For a given L_Δ, we can obtain an HSM with the same decomposability as L_Δ
- Next we discuss decomposability of a given log-affine model L
- Every log affine model has a hierarchical model for which L is an HSM
- From a viewpoint of localization of the inference, a natural question is to look for a small L_Δ for which L is an HSM
- We derive the smallest decomposable model for which L is an HSM

$\Downarrow$

Ambient decomposable model

Ambient decomposable model

Decomposition of log-affine model

- In the case of the hierarchical model
  interaction terms $\Leftrightarrow \Delta$
  decomposition of $L_\Delta \Leftrightarrow$ decomposition of $\mathrm{red}\Delta$ (hypergraph)
- L is not necessarily an HSM of a hierarchical model which has the same interaction as L
  ex) 3-way conditional independence model
  - $L_\Delta = L_{\{1,2\}} + L_{\{2,3\}}$
    $$\log p_{ijk} = \alpha_i + \beta_j + \gamma_k + \delta_{ij} + \delta'_{jk}$$
  - L: $\log p_{ijk} = \alpha_i + \beta_j + \gamma_k + \delta\phi_{ij} + \delta'\psi_{jk}$
  - L': $\log p_{ijk} = \alpha_i + \beta_j + \gamma_k + \delta(\phi_{ij} + \psi_{jk})$
- Decomposition of L does not always correspond to simplicial complex induced by interaction terms
- We define the decomposition of L by using the conformality

Hypergraph induced by HSM

- $S \subset [m]$ is a partial edge separator (pes) of L if
  - $L_S \subset L$
  - For disjoint subsets $A_1 \cup A_2 \cup S = [m]$,
    L is conformal to $\{L_{A_1 \cup S}, L_{A_2 \cup S}\}$
- (A_1, A_2, S) is called a decomposition of L
- u and v are tightly connected in L if there is no pes of L s.t. $u \in A_1$ and $v \in A_2$
- Extended compact component (ECC)
  a set of vertices any two of which are tightly connected in L
- Hypergraph $\mathcal{H}$ induced by L
  the set of maximal ECCs

Ambient decomposable model

- $L_{\mathcal{H}}$: the hierarchical model induced by $\mathcal{H}$

Theorem

$L_{\mathcal{H}}$ is the ambient decomposable model of L

- MLE

$$\hat{p}(\boldsymbol{i}) = \frac{\prod_{C\in\mathcal{H}} \hat{p}(\boldsymbol{i}_C)}{\prod_{S\in\mathcal{S}} \hat{p}(\boldsymbol{i}_S)} = \frac{\prod_{C\in\mathcal{H}} \hat{p}(\boldsymbol{i}_C)}{\prod_{S\in\mathcal{S}} x(\boldsymbol{i}_S)/n}.$$

  - $\mathcal{S}$: the set of divider of $\mathcal{H}$
  - $\hat{p}(\boldsymbol{i}_C)$ depends only on the marginal table $x(\boldsymbol{i}_C)$

$\Downarrow$

The computation of the MLE is localized to ECCs

Ambient decomposable model

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- MLE

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Local computation of Markov bases

- Dobra and Sullivant (2004)
  A Markov basis of a hierarchical model L_Δ is computed recursively from Markov bases of marginal models L_C for all $C \in \mathcal{C}$
- A Markov basis of an HSM is also computed from Markov bases of $L \cap L_C$, $C \in \mathcal{H}$

Local computation of Markov bases

- (A_1, A_2, S) : decomposition of L
- $\boldsymbol{z}_1$: a move $L \cap L_{A_1 \cup S}$

$$\boldsymbol{z}_1 = [\{(\boldsymbol{i}_1, \boldsymbol{j}_1), \ldots, (\boldsymbol{i}_d, \boldsymbol{j}_d)\} || \{(\boldsymbol{i}'_1, \boldsymbol{j}_1), \ldots, (\boldsymbol{i}'_d, \boldsymbol{j}_d)\}],$$

$$\boldsymbol{i}_k, \boldsymbol{i}'_k \in \mathcal{I}_{A_1}, \quad \boldsymbol{j}_k \in \mathcal{I}_S$$

  - $(\boldsymbol{i}_1, \boldsymbol{j}_1), \ldots, (\boldsymbol{i}_d, \boldsymbol{j}_d)$:
    cells (with replication) of "+" elements of z_1
  - $(\boldsymbol{i}'_1, \boldsymbol{j}_1), \ldots, (\boldsymbol{i}'_d, \boldsymbol{j}_d)$:
    cells (with replication) of "$-$" elements of z_1
  - $L_S \subset L \Rightarrow z_1(\boldsymbol{i}_S) = 0$

Extension of moves

Definition $(\mathrm{Ext}(\mathcal{B}(A_1 \cup S) \to L))$

$V_1 := A_1 \cup S$
$\mathcal{B}(V_1)$: a Markov basis of $L \cap L_{V_1}$
$\boldsymbol{z}_1 \in \mathcal{B}(V_1)$
$\boldsymbol{k} := \{\boldsymbol{k}_1, \dots, \boldsymbol{k}_d\} \in \mathcal{I}_{A_2} \times \cdots \times \mathcal{I}_{A_2}$,
Define $\boldsymbol{z}_1^{\boldsymbol{k}}$ by

$$\boldsymbol{z}_1^{\boldsymbol{k}} := [\{(\boldsymbol{i}_1, \boldsymbol{j}_1, \boldsymbol{k}_1), \dots, (\boldsymbol{i}_d, \boldsymbol{j}_d, \boldsymbol{k}_d)\} || \{(\boldsymbol{i}'_1, \boldsymbol{j}_1, \boldsymbol{k}_1), \dots, (\boldsymbol{i}'_d, \boldsymbol{j}_d, \boldsymbol{k}_d)\}].$$

Then define $\mathrm{Ext}(\mathcal{B}(V_1) \to L)$ by

$$\mathrm{Ext}(\mathcal{B}(V_1) \to L) := \{\boldsymbol{z}_1^{\boldsymbol{k}} \mid \boldsymbol{k} \in \mathcal{I}_{A_2} \times \cdots \times \mathcal{I}_{A_2},\ \boldsymbol{z}_1 \in \mathcal{B}(V_1)\}$$

$$\begin{array}{cccc} i_1j_1 & i'_1j_1 & i_2j_2 & i'_2j_2 \\ \hline 1 & -1 & 1 & -1 \end{array} \quad \Rightarrow \quad \begin{array}{c|cccc} & i_1j_1 & i'_1j_1 & i_2j_2 & i'_2j_2 \\ \hline k_1 & 1 & -1 & 0 & 0 \\ k_2 & 0 & 0 & 1 & -1 \\ \hline & 1 & -1 & 1 & -1 \end{array}$$

Local computation of Markov bases

Theorem

$V_1 := A_1 \cup S$, $V_2 := A_2 \cup S$
$\mathcal{B}(V_1)$, $\mathcal{B}(V_2)$: MB of $L \cup L_{V_1}$, $L \cup L_{V_2}$
$\mathcal{B}_{V_1,V_2}$: a MB of decomposable model with two cliques V_1, V_2

Then

$$\mathcal{B} := \mathrm{Ext}(\mathcal{B}(V_1) \to L) \cup \mathrm{Ext}(\mathcal{B}(V_2) \to L) \cup \mathcal{B}_{V_1,V_2}$$

is a Markov basis of L

Numerical example

Numerical example : WAM data

- Woman and Mathematics (WAM) data : 6-way contingency table
  a questionnaire from high school students in NJ
  (Fowlkes(1988),Højsgaard(2003))

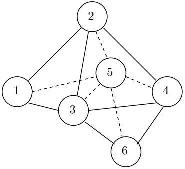
(1) Attendance in math lectures (attended=1, not=2)
(2) Sex (female=1, male=2)
(3) School type (suburban=1, urban=2)
(4) Agree in statement "I'll need mathematics in my future work" (agree=1, disagree=2)
(5) Subject preference (math-science=1, liberal arts=2)
(6) Future plans (college=1, job=2)

1. H_1 : decomposable model

$$L_1 \ = \ L_{\{1,2,3,5\}} + L_{\{2,3,4,5\}} + L_{\{3,4,5,6\}}$$

2. H_0 : split model

$$L_0 \ = \ L_{\{1,2,3,5\}} + L^{j_3=1}_{\{2,5\}} + L^{j_3=1}_{\{4,5\}} + L^{j_3=2}_{\{2,4,5\}} + L_{\{3,4,5,6\}}.$$



1.	attendance in math	4.	necessity of math in future
2.	sex	5.	subject preference
3.	school type	6.	future plan

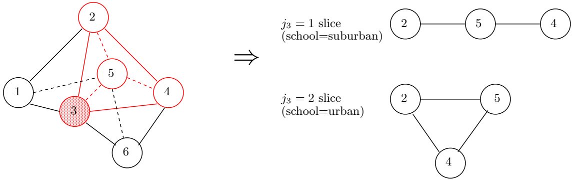
Numerical example : model

1. H_1 : decomposable model

$$L_1 \;=\; L_{\{1,2,3,5\}} + L_{\{2,3,4,5\}} + L_{\{3,4,5,6\}}$$

2. H_0 : split model

$$L_0 \;=\; L_{\{1,2,3,5\}} + L^{j_3=1}_{\{2,5\}} + L^{j_3=1}_{\{4,5\}} + L^{j_3=2}_{\{2,4,5\}} + L_{\{3,4,5,6\}}.$$



1.	attendance in math	4.	necessity of math in future
2.	sex	5.	subject preference
3.	school type	6.	future plan

Numerical example : Markov basis

- $\mathcal{B}_1$: Markov basis of L_1

$$\mathcal{B}_1 \;=\; \mathcal{B}_{\{1,2,3,5\},\{2,3,4,5,6\}} \cup \mathcal{B}_{\{1,2,3,4,5\},\{3,4,5,6\}}$$

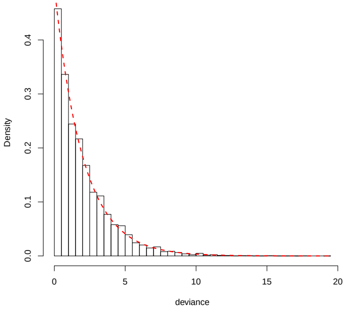
- $\mathcal{B}_0$: Markov basis of L_0

$$\mathcal{B}_0 \;=\; \mathcal{B}^{i_3=1}_{\{1,2,5\},\{4,5,6\}} \cup \mathcal{B}_1$$

  - $\mathcal{B}_{C_1,C_2}$: a Markov basis of a decomposable model with two cliques C_1 and C_2
  - $\mathcal{B}^{i_\delta}_{\mathcal{C}}$: $\mathcal{B}_{\mathcal{C}}$ on i_δ-slice

Numerical example : results

- We used LR statistic as a test statistic



	p-value	
Deviance	asymptotic χ^2_2	MCMC
1.851	0.396	0.399$\pm$0.012

- split model is accepted

- histgram : MCMC
- dotted line : asymptotic χ^2_2

Summary and future work

Summary

- We proposed a hierarchical subspace model by defining the notion of conformality of linear subspaces to a given hierarchical model
- The notion of an HSM gives a modeling strategy of multiway tables and unifies various models of interaction effects
- We illustrated practical advantage of our modeling strategy with some data sets

Future work

- Is it possible to treat nonlinear models such as the RC association model in the framework of HSM ?

References

Hara, H., Sei, T. and Takemura, A.(2009).
Hierarchical subspace models for contingency tables
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