# Commuting Birth-Death Processes

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# Birth-Death Process on a 2-dim'l Grid

Consider a discrete-time Markov chain $Z$ with state space $E = \{0, 1, \ldots, m\} \times \{0, 1, \ldots, n\}$. The chain makes jumps of size one either upwards, downwards, to the right, or to the left.

We regard $E$ as a graph in the plane, called the *$m \times n$ grid*. The dynamics of $Z$ are specified by the transition probabilities

$$
\begin{aligned}
L_{ij} &:= \mathrm{Prob}\{Z_{k+1} = (i-1, j) \mid Z_k = (i, j)\}, \\
R_{ij} &:= \mathrm{Prob}\{Z_{k+1} = (i+1, j) \mid Z_k = (i, j)\}, \\
D_{ij} &:= \mathrm{Prob}\{Z_{k+1} = (i, j-1) \mid Z_k = (i, j)\}, \\
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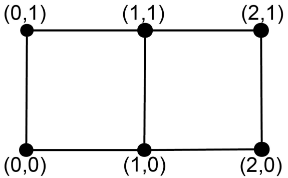
The transition matrix $P$ of the Markov chain $Z$ has format

$$(m+1)(n+1) \times (m+1)(n+1).$$

We write it as a sum of two matrices, $P = P_h + P_v$, one for horizontal moves and one for vertical moves.

# The 2-by-1 Grid




$P_h =$

| | $(0,0)$ | $(0,1)$ | $(1,0)$ | $(1,1)$ | $(2,0)$ | $(2,1)$ |
|---|---|---|---|---|---|---|
| $(0,0)$ | 0 | 0 | $R_{00}$ | 0 | 0 | 0 |
| $(0,1)$ | 0 | 0 | 0 | $R_{01}$ | 0 | 0 |
| $(1,0)$ | $L_{10}$ | 0 | 0 | 0 | $R_{10}$ | 0 |
| $(1,1)$ | 0 | $L_{11}$ | 0 | 0 | 0 | $R_{11}$ |
| $(2,0)$ | 0 | 0 | $L_{20}$ | 0 | 0 | 0 |
| $(2,1)$ | 0 | 0 | 0 | $L_{21}$ | 0 | 0 |

$P_v =$

| | $(0,0)$ | $(0,1)$ | $(1,0)$ | $(1,1)$ | $(2,0)$ | $(2,1)$ |
|---|---|---|---|---|---|---|
| $(0,0)$ | 0 | $U_{00}$ | 0 | 0 | 0 | 0 |
| $(0,1)$ | $D_{01}$ | 0 | 0 | 0 | 0 | 0 |
| $(1,0)$ | 0 | 0 | 0 | $U_{10}$ | 0 | 0 |
| $(1,1)$ | 0 | 0 | $D_{11}$ | 0 | 0 | 0 |
| $(2,0)$ | 0 | 0 | 0 | 0 | 0 | $U_{20}$ |
| $(2,1)$ | 0 | 0 | 0 | 0 | $D_{21}$ | 0 |

# Commuting Variety

We are interested in those Markov chains for which the horizontal matrix $P_h$ commutes with the vertical matrix $P_v$. This happens if and only if the following four constraints hold for all index pairs:

$$\begin{aligned}
U_{i,j}R_{i,j+1} &= R_{i,j}U_{i+1,j} && \text{(up-right)}\\
D_{i,j+1}R_{i,j} &= R_{i,j+1}D_{i+1,j+1} && \text{(down-right)}\\
D_{i+1,j+1}L_{i+1,j} &= L_{i+1,j+1}D_{i,j+1} && \text{(down-left)}\\
U_{i+1,j}L_{i+1,j+1} &= L_{i+1,j}U_{i,j} && \text{(up-left).}
\end{aligned}$$

In English: for each corner of a square in the grid, the probability of going from that corner to the diagonally opposite corner of the square in two steps is the same for the two possible paths.

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In English: for each corner of a square in the grid, the probability of going from that corner to the diagonally opposite corner of the square in two steps is the same for the two possible paths.

Let $I^{(m,n)}$ be the binomial ideal generated by these constraints.

The analogous birth-death process on the $d$-dimensional grid of format $n_1 \times \cdots \times n_d$ also defines a binomial ideal $I^{(n_1,\ldots,n_d)}$.

# Primary Decomposition for the 2-by-1 Grid

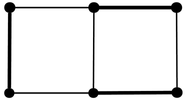
The ideal $I^{(2,1)}$ is radical. Its 11 components come in six classes:

1. The main component has codimension 6 and degree 16:

$$I_{\mathcal{A}} = \mathbb{I}_2\begin{pmatrix} R_{00} & U_{00} & L_{11} & U_{11} \\ R_{01} & U_{10} & L_{10} & D_{01} \end{pmatrix} + \mathbb{I}_2\begin{pmatrix} R_{10} & U_{10} & L_{21} & D_{21} \\ R_{11} & U_{20} & L_{20} & D_{11} \end{pmatrix}$$

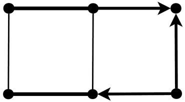
2. Two components of codimension 7 and degree 4 $\simeq V(I^{(1,1)})$
3. The monomial component representing horizontal transitions has codimension six: $\langle\, U_{00},\ U_{10},\ U_{20},\ D_{01},\ D_{11},\ D_{21}\,\rangle$
4. The monomial component representing vertical transitions has codimension eight: $\langle\, R_{00}, R_{01}, R_{10}, R_{11}, L_{10}, L_{11}, L_{20}, L_{21}\,\rangle$
5. Another pair of monomial primes of codimension 8, such as

$$\langle\, R_{00}, R_{01}, L_{10}, L_{11}, U_{10}, U_{20}, D_{11}, D_{21}\,\rangle$$



6. Four monomial primes of codimension 7, such as

$$\langle\, R_{10},\ L_{21},\ U_{00},\ U_{10},\ D_{01},\ D_{11},\ D_{21}\,\rangle.$$



# The 3-Cube and a Conjecture

$I^{(1,1,1)}$ is generated by 24 quadratic binomials (four for each face of the 3-cube) in 24 unknowns (two for each edge of the 3-cube):

$$\begin{gathered}\langle\, L_{100}U_{000}-U_{100}L_{110},\ L_{101}U_{001}-U_{101}L_{111},\ L_{110}D_{010}-D_{110}L_{100},\ L_{111}D_{011}-D_{111}L_{101},\\ L_{100}F_{000}-F_{100}L_{101},\ L_{110}F_{010}-F_{110}L_{111},\ L_{101}B_{001}-B_{101}L_{100},\ L_{111}B_{011}-B_{111}L_{110},\\ R_{000}U_{100}-U_{000}R_{010},\ R_{001}U_{101}-U_{001}R_{011},\ R_{010}D_{110}-D_{010}R_{000},\ R_{011}D_{111}-D_{011}R_{001},\\ R_{000}F_{100}-F_{000}R_{001},\ R_{010}F_{110}-F_{010}R_{011},\ R_{001}B_{101}-B_{001}R_{000},\ R_{011}B_{111}-B_{011}R_{010},\\ B_{101}U_{100}-U_{101}B_{111},\ B_{001}U_{000}-U_{001}B_{011},\ B_{011}D_{010}-D_{011}B_{001},\ B_{111}D_{110}-D_{111}B_{101},\\ F_{000}U_{001}-U_{000}F_{010},\ F_{100}U_{101}-U_{100}F_{110},\ F_{010}D_{011}-D_{010}F_{000},\ F_{110}D_{111}-D_{110}F_{100}\,\rangle.\end{gathered}$$

This ideal is the intersection of 135 prime ideals. The main component is a unimodular toric ideal $I_{\mathcal{A}}$ of codimension 14 and degree 300. Its Markov basis has 53 elements (33 quadrics, 8 cubics and 12 quartics), and its Graver basis has 3698 elements (42 quadrics, 224 cubics, 1032 quartics, 1728 quintics and 672 sextics).

Besides $I_{\mathcal{A}}$, there are 91 monomial primes and 43 other primes.

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B_{101}U_{100} - U_{101}B_{111},\; B_{001}U_{000} - U_{001}B_{011},\; B_{011}D_{010} - D_{011}B_{001},\; B_{111}D_{110} - D_{111}B_{101}, \\
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### Conjecture (disproved by Thomas Kahle)

*The binomial ideal $I^{(n_1,\dots,n_d)}$ is always radical.*

# Parametrization of the Main Component

**Theorem**

*Fix strictly positive transition probabilities $P(u,v)$ on a $d$-dimensional grid. The directional transition matrices $P_1, P_2, \ldots, P_d$ commute pairwise if and only if*

$$P(u,v) \;=\; a_u \cdot W(u,v) \cdot a_v^{-1}, \qquad u,v \in E,$$

*for some collection of parameters $a_u$ and $W(u,v)$ that satisfy $W(u',v') = W(u'',v'')$ when $v'-u' \in \{\pm e_k\}$ for some $k$ and $\{u'',v''\} = \{u'+w, v'+w\}$ for some $w \in \sum_{\ell \neq k} \mathbb{Z} e_\ell$. The $a_u$ are unique up to a common multiple, and the $W(u,v)$ are unique.*

We write this parametrization as an integer matrix $\mathcal{A}$ with entries $0, +1, -1$. The corresponding toric ideal $I_{\mathcal{A}}$ is our main component.

# The Integer Matrix $\mathcal{A}$

Let $\mathcal{A}^{(n_1,\dots n_d)}$ be a matrix that has one column for each grid edge $(u,v)$ and one row for each parameter $a_u$ or $W(u,v)$. The column for $(u,v)$ has $+1$ in the rows $a_u$ and $W(u,v)$, and $-1$ in row $a_v$.

$$\text{Format}: \qquad \left(\prod_{i=1}^{m}(n_i+1)+\sum_{i=1}^{m} n_i\right)\times\left(2\sum_{i=1}^{m} n_i \prod_{j\neq i}(n_j+1)\right).$$

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Example: The $9\times 14$-matrix $\mathcal{A}^{(2,1)}$ has rank 8 and equals

| | $R_{00}$ | $R_{01}$ | $R_{10}$ | $R_{11}$ | $L_{10}$ | $L_{11}$ | $L_{20}$ | $L_{21}$ | $U_{00}$ | $U_{10}$ | $U_{20}$ | $D_{01}$ | $D_{11}$ | $D_{21}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $a_{00}$ | 1 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 1 | 0 | 0 | -1 | 0 | 0 |
| $a_{01}$ | 0 | 1 | 0 | 0 | 0 | -1 | 0 | 0 | -1 | 0 | 0 | 1 | 0 | 0 |
| $a_{10}$ | -1 | 0 | 1 | 0 | 1 | 0 | -1 | 0 | 0 | 1 | 0 | 0 | -1 | 0 |
| $a_{11}$ | 0 | -1 | 0 | 1 | 0 | 1 | 0 | -1 | 0 | -1 | 0 | 0 | 1 | 0 |
| $a_{20}$ | 0 | 0 | -1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | -1 |
| $a_{21}$ | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 1 | 0 | 0 | -1 | 0 | 0 | 1 |
| $h_1$ | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $h_2$ | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $v_1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |

# Unimodularity and Cayley Structure

The matrix $\mathcal{A}$ is a *Cayley matrix*, i.e., it has a block structure like

$$\begin{bmatrix} \boxed{\quad * \quad} & \boxed{\quad * \quad} & \boxed{\; * \;} & \boxed{\quad * \quad} \\ 1 \cdots 1 & 0 & 0 & 0 \\ 0 & 1 \cdots 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \cdots 1 \end{bmatrix}.$$

An integer matrix of rank $r$ is *unimodular* if all its non-zero $r \times r$ minors have the same absolute value. Investigating whether a matrix is unimodular is interesting from the perspectives of algebraic statistics, toric algebra and integer programming.

**Theorem**

*The Cayley matrix $\mathcal{A} = \mathcal{A}^{(n_1,\dots,n_d)}$ is unimodular if and only if $d = 2$ and the format of the grid is either $2 \times 2$ or $n \times 1$.*

# Conclusions

This talk concerned the multilinear algebra problem of characterizing certain pairs of commuting matrices.

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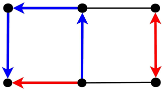

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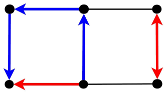

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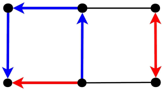

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**Many thanks to Rudy Yoshida for organizing this session.**